Iterative Solvers For Singular Symmetric Linear Systems in Low Frequency Electromagnetics

A. Tinzefte, Y. Le Menach and F. Piriou
LAMEL, L2EP, USTL, Cité Scientifique 59655 Villeneuve d'Ascq, France

***Abstract*—In this paper, several methods based on Krylov methods are proposed to solve the singular linear systems from Finite Element Method. Indeed, in the magnetostatic case, for A-formulation the system to solve is singular but it is auto-gauged by Krylov methods. However, due to the computation of residual vector all the methods (CG, MRTR, SQMR,MINRES) do not present the same behaviour. Moreover, these methods are applied to eddy current problem. The numerical behavior are compared and analyzed.**

***Index Terms*—Numerical analysis, Eddy currents, Magnetostatics.**

I. Introduction

For the Finite Element Method (FEM) applied to electromagnetism, we are led to solve linear system of equations, there are different methods for solving efficaciously such systems. Given sufficient memory and reasonable performance, direct solvers would be the method of choice for solving linear systems which have a regular matrix. The Krylov subspace iterative methods such as Conjugate Gradient (CG) [7] require less memory than direct solvers. Consequently, they are preferred for very large sparse linear systems that cannot fit into memory using direct solvers. Also, these methods represent a good computation tool for solving singular systems. This is the case in magnetostatic when we use the formulation in term of the vector potential without gauge condition. The gauge condition can be introduced with the help of an edge tree; alternative gauges are possible. With the gauge condition the system is well defined but the convergence of iterative method is very slow, so the gauging is impracticable [10].

In the case of non-gauged formulation, the properties of iterative solvers can be used to auto-gauged the problem [10]. So, the system converges and has an infinity of solutions (potential) if the matrix equation is compatible. In the case of vector formulation (curl-curl equation), the compatibility implies that the right hand side (the current density) should be divergence free. We can see, that all solutions in term of potential give the same magnetic field. The same finding is observed for eddy current problem, when we use the potential formulations.

The CG is reliable on positive-definite systems. Although, we notice that, in the case of singular systems, this method can breakdown and his convergence is irregular. In this paper, we propose to use other methods of Krylov subspace type, which have a much more stable behaviour. An example is proposed to illustrate our results.

A. Tinzefte, Y. Le Menach and F. Piriou, are with the Laboratory of Electrotechnics and Power Electronics (L2EP), University of Lille, 59655 Villeneuve dAscq cedex, France (e-mail: yvonnick.le-menach@univ-lille1.fr).

II. Model

A. magnetostatic Case

The magnetostatic assumptions introduced into Maxwell's equation and with the help of vector potential and the magnetic relationship, we obtain the classical A-formulation.

$$\mathbf{curl}\frac{\mathbf{1}}{\mu}\mathbf{curlA} = \mathbf{J} \tag{1}$$

With μ the magnetic permeability, $\mathbf{J}$ the current density and $\mathbf{A}$ the magnetic vector potential. According to the classical works [2] [14] concerning the discretisation of magnetic quantities, the vector potential $\mathbf{A}$ is defined on the edges (circulation) and the current density $\mathbf{J}$ on the facets (flux). But to define the discrete form of the constitutive law we must introduce the duality conditions between two series of spaces. In these conditions, if we consider a primal mesh on which the vector potential (noted $\mathbf{A}_e$) then the current density (noted $\widetilde{\mathbf{J}}_f$) will be expressed on facets of the dual mesh $\widetilde{\mathbf{M}}$. Moreover, using the properties of the incidence matrix we can define the discrete operators of gradient, curl and divergence. These operators are denoted on the primal mesh: $\mathbf{G}$, $\mathbf{C}$ and $\mathbf{D}$ and respectively $\widetilde{\mathbf{G}}$, $\widetilde{\mathbf{C}}$ and $\widetilde{\mathbf{D}}$ on the dual mesh. It can be noted that we have the following properties: $\mathbf{G}^T = -\widetilde{\mathbf{D}}$ and $\mathbf{C} = \widetilde{\mathbf{C}}^T$. Consequently, the discretized form of $\mathbf{A}$ formulation (1) can be written

$$\mathbf{C^T M^{\nu}_{ff} C A_e} = \widetilde{\mathbf{J}}_\mathbf{f} \tag{2}$$

where $\mathbf{M^{\nu}_{ff}}$ is the material matrix including the permeability and the metric information. Usually, the current density are known on primal mesh then a matrix (noted $\mathbf{M_{ef}}$) containing only geometric data must be introduced to transfer the current density on dual mesh.

$$\mathbf{C^T M^{\nu}_{ff} C A_e} = \mathbf{M_{ef} J_f} \tag{3}$$

The system is singular. Indeed an infinity of solution can verify the equation (1) if $\mathbf{M_{ef} J_f}$ is in the range of $\mathbf{C^T M^{\nu}_{ff} C}$ (divergence of $\mathbf{J}$ must be free [1]). In fact, the system is oversized by the number of nodes minus one equation. To ensure the uniqueness of $\mathbf{A_e}$ there are two possibilities : the first one consists to use a spanning edge tree to eliminate the useless unknowns. The second one consists to impose the new constraint on the $\mathbf{A_e}$ adding the coulomb gauge $\widetilde{\mathbf{D}}\mathbf{A_e} = 0$. This gauge can be write such that $-\mathbf{G}^T\mathbf{A_e} = 0$. As the size of $\mathbf{G}^T$ is the product of the node number by edge number, and to ensure the solution one potential must fixed. Consequently, the system becomes well-defined by adding this gauge condition.

In the case of non-gauged formulation, Z. Ren [10] clarified the coulomb gauge are weakly imposed with the algorithm of conjugate gradient (CG).

B. eddy current problem

Many formulations can be used to compute the eddy current (**H**-formulation, **T**-Ω formulation and **A**-φ formulation). In this paper, the potential vector introduced in the magnetostatic case is kept. Then we present only the **A**-φ formulation. In Maxwell-Ampere equation, the current density represents the eddy current. In this conditions, with the help of the electrical scalar potential the equation (3) becomes

$$\mathbf{curl}\frac{1}{\mu}\mathbf{curlA} + \sigma(\frac{\partial \mathbf{A}}{\partial t} + \mathbf{grad}\varphi) = \mathbf{0} \tag{4}$$

Moreover, the conservation of the eddy current must verified.

$$\mathbf{div}\sigma(\frac{\partial \mathbf{A}}{\partial t} + \mathbf{grad}\varphi) = \mathbf{0} \tag{5}$$

The discretisation can be carried out with the same scheme which present in the previous paragraph.

$$\begin{aligned} \mathbf{C^T M^{\nu}_{ff} C A_e} + \mathbf{M^{\sigma}_{ee}}(\frac{\partial \mathbf{A_e}}{\partial \mathbf{t}} + \mathbf{G}\varphi_\mathbf{n}) = 0 \\ \mathbf{G^T M^{\sigma}_{ee}}(\frac{\partial \mathbf{A_e}}{\partial \mathbf{t}} + \mathbf{G}\varphi_\mathbf{n}) = 0 \end{aligned} \tag{6}$$

where $\mathbf{M^{\sigma}_{ee}}$ is the material matrix including the conductivity and the metric information. This system is singular; indeed the number of these unknowns is the sum of the nodes and edges, and the rank of the system matrix is equal to the number of edge. A gauge condition must be applied to obtain the uniqueness of the solution. Although, the auto-gauged of the CG is not proved for **A**-φ formulation.

III. Iterative Methods

In the previous section, we have to solve a system of linear equations

$$Ax = b, \tag{7}$$

where $A \in I\!R^{N\times N}$ is symmetric semi-definite, x and $b \in I\!R^N$. Two cases can be distinguished:

1) The case where A is nonsingular and consequently positive definite.
2) The case where A is singular.

Many results are known about the first case [7], where the CG is a successful iterative method.
We are interested by the second case where A is singular, here, the system (7) has a solution if, and only if, b is in the range of A. In that case the solution is not unique. Indeed, Let $x \in I\!R^N$ be a solution of (7), then $\hat{x} = x + y$ is a solution for every y in the Kernel space of A.

An important class of iterative methods available for solving the system (7) are the so-called Krylov subspace methods.
An iterative scheme for solving linear systems (7) is called a Krylov subspace method if it produces approximate solutions of the form

$$x_k \in x_0 + \mathcal{K}_k(A, r_0), \quad k = 1, 2, \ldots \tag{8}$$

where x_0 is an arbitrary initial guess with the corresponding residual vector $r_0 = b - Ax_0$ and $\mathcal{K}_k(A, r_0)$ is the kth Krylov subspace defined by

$$\mathcal{K}_k(A, r_0) = span\{r_0, Ar_0, \ldots, A^{k-1}r_0\} \tag{9}$$

There are two main steps in designing a Krylov subspace method. The first step is the construction of suitable vectors $v_1, v_2, \ldots, v_k$ that span $\mathcal{K}_k(A, r_0)$. Then, setting

$$V_k = [v_1, v_2, \ldots, v_k] \tag{10}$$

we parameterize the kth iteration (8) as follows :

$$x_k = x_0 + V_k z_k, \quad \text{where} \quad z_k \in I\!R^k \tag{11}$$

Therefore, as a second step, it remains to specify the choice of z_k in (11). Various strategies in the above two steps lead to various Krylov subspace methods. When A is nonsymmetric, there are Krylov subspace type iterative solvers based on biorthogonality, such as the BICG method [4] and its modified versions such as the CGS [13], Bicg-stab [15] and QMR [5] methods. There are also methods based on minimizing the residual $r = b - Ax$, such as the Generalized Conjugate Residual (GCR) method [3] and the Generalized Minimum Residual (GMRES) method [11].

This paper is devoted to a symmetric systems. We present some Krylov subspace type iterative methods adapted for this case. The CG is the most popular one. It is effective for systems of the form (7) where A is symmetric positive definite. The CG algorithm can be derived from the Lanczos process who construct a basis vectors $\{v_j\}$ for $\mathcal{K}_k(A, r_0)$ and the vector z_k is chosen such as the residual r_k verifies the Galerkin condition

$$r_k \perp \mathcal{K}_k(A, r_0) \quad \Leftrightarrow \quad V_k^T r_k = 0. \tag{12}$$

The Lanczos process iteratively computes vectors v_k as follows: $v_0 = 0, \quad \beta_1 v_1 = r_0$ where $\beta_1 = \|r_0\|$,

$$\alpha_k = v_k^T A v_k, \quad \beta_{k+1} v_{k+1} = A v_k - \alpha_k v_k - \beta_k v_{k-1}$$

with $\beta_{k+1} \geqq 0$ chosen so that $\|v_{k+1}\| = 1$. The Lanczos process transforms a symmetric matrix A to a symmetric tridiagonal matrix :

$$T_k := \begin{bmatrix} \alpha_1 & \beta_2 & 0 & \cdots & 0 \\ \beta_2 & \alpha_2 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \beta_k \\ 0 & \cdots & 0 & \beta_k & \alpha_k \end{bmatrix}.$$

In exact arithmetic, the process terminates after $m \leq N$ steps, *i.e.* $\beta_k \neq 0$ for $k = 1, \ldots, m$ and $\beta_{m+1} = 0$. So from now on we can assume that $\beta_k \neq 0, \quad k = 1, \ldots, m$.
After the kth step, we have

$$\begin{aligned} AV_k &= V_k T_k + \beta_{k+1} v_{k+1} {e_k^{(k)}}^T \\ V_k^T V_k &= I = [e_1^{(k)}, \ldots, e_k^{(k)}], \quad V_k^T v_{k+1} = 0 \end{aligned} \tag{13}$$

then $V_k^T A V_k = T_k$ and $V_k^T r_0 = \beta_1 e_1^{(k)}$.
So, (11) becomes $r_k = r_0 - A V_k z_k$. And with the choice (12), we have

$$V_k^T r_k = V_k^T r_0 - V_k{}^T A V_k z_k = 0$$

and finally we obtain

$$T_k z_k = \beta_1 e_1^{(k)}.$$

The CG attempt to solve this system by applying the Cholesky decomposition to T_k. If A is positive definite then so is T_k, and hence the Cholesky decomposition

$$T_k = L_k D L_k^T$$

exists. Here D_k is diagonal with positive element, and L_k is unit lower bidiagonal. If A is an indefinite symmetric matrix, then the Cholesky factorisation of T_k can still be tried, often with success, but it does not always exist and can no longer be relied upon numerically, and the CG became instable numerically. Otherwise, CG may breakdown when $\alpha_k = v_k^T A v_k = 0$ while $v_k \neq 0$, here the algorithm is stopping prematurely. This is also the case when the matrix A is singular.
Because of the problems cited above, the CG is impracticable for singular systems. That is why we propose the use of the following methods which turn out more stable numerically.

MINRES (Minimum Residual Method [9]), SQMR (Symmetric quasi-minimal residual method [6]) and MRTR (Minimized Residual method based on the three-term recurrence formula of CG type [1]) are Krylov subspace methods for solving large symmetric systems, these methods are well adapted for the indefinite systems. they are based on the Lanczos process to construct a basis of $\mathcal{K}_k(A, r_0)$.
MINRES, SQMR and MRTR replace the Galerkin condition by the residual minimisation :

$$\|r_k\|_2 = \min_{x \in x_0 + \mathcal{K}_k(A, r_0)} \|b - Ax\|_2$$

By using (11) and (13), we can write

$$\begin{aligned} r_k &= b - Ax_k = b - Ax_0 - AV_k z_k = r_0 - V_{k+1}\tilde{T}_k z_k \\ &= \beta_1 v_1 - V_{k+1}\tilde{T}_k z_k = V_{k+1}(\beta_1 e_1^{k+1} - \tilde{T}_k z_k) \\ \Rightarrow \|r_k\|_2 &= \|\beta_1 e_1^{k+1} - \tilde{T}_k z_k\|_2 \end{aligned}$$

Since V_{k+1} has orthonormal columns. With $\|.\|_2$ the related Euklidean norms and

$$\tilde{T}_k = \begin{bmatrix} T_k \\ 0 \cdots 0 \ \beta_{k+1} \end{bmatrix}$$

Therefore, MINRES and SQMR characterize the kth approximate solution as $x_k = x_0 + V_k z_k$, where $z_k \in \mathbb{R}^k$ is a solution of the least-squares problem

$$\|r_k\|_2 = \|\beta_1 e_1^{k+1} - \tilde{T}_k z_k\|_2 = \min_{y \in \mathbb{R}^k} \|\beta_1 e_1^{k+1} - \tilde{T}_k y\|_2$$

Next, for solving this problem, we use a QR-decomposition of $\tilde{T}_k$ by means of plane rotations called Givens rotations.

Let us note that the SQMR is an adaptation of the generale nonsymmetric QMR method to exploit the symmetric of the matrix A. In the implementation of the nonsymmetric QMR, z_k is chosen as the solution of the least-squares problem

$$\|\beta_1 e_1^{k+1} - \tilde{T}_k z_k\|_2 = \min_{y \in \mathbb{R}^k} \|\beta_1 e_1^{k+1} - \tilde{T}_k y\|_2$$

even if V_{k+1} is not orthonormal in the nonsymmetric case. This is called a *quasi-minimisation residual* property.

In the MRTR, the Lanczos process is implemented differently than the classical one. And the residual minimization is used to determine the parameters of the residual, see [1].

When the system is singular, the CG method and methods based on biorthogonality may diverge when T_k is singular. An otherwise, for methods based on minimizing the residual, as shown in [12], the coefficients s_k and c_k of the Givens rotation matrix verify $\|r_k\|_2 = |s_k| \|r_{k-1}\|_2$ with $c_k + s_k = 1$; and $s_k = 1$ is equivalent to $det(T_k) = 0$.
So, at the kth step, there will be stagnation of the residual norm associated with the solution x_k if and only if $s_k = 1$ (T_k singular). Therefore, in exact arithmetic, the residual is expected to decrease monotonically without divergence and stagne if T_k is singular.

The convergence rate of all these methods is determined by the spectrum of eigenvalues of the matrix A. An acceleration of the convergence rate can often be achieved by replacing the system (7) by the preconditioned system.

$$M^{-1} A x = M^{-1} b$$

The matrix M must be chosen nonsingular and in such a way that the system $Mz = y$ is much easier to solve than the original system for every vector y on the right hand side of the equation, and also, so that the matrix $M^{-1}A$ has a more 'favorable' spectrum of eigenvalues than A.

Let us note that both CG and MINRES require a symmetric positive definite preconditioners while with MRTR and SQMR one can use symmetric indefinite preconditioners for symmetric but indefinite systems.

There are several technique to construct a preconditioners for the symmetric system. The incomplete Cholesky decomposition (IC) [7] is a good choice if A is symmetric definite positive. But when A is semi-definite or indefinite, IC may not exist. The SSOR is also a good choice and exist if the diagonal of A is nonzero.

IV. Applications

As example of magnetostatic problem, we propose to study with **A** formulation a Three-phase transformer which the mesh is presented on the figure (1.a). A current density is applied in only one winding. To ensure the divergence free a facet tree technique is used [8]. The flux density in the iron core is shown on the figure (1.b). In the case of non-gauged formulation, we obtain a singular system which have 49558 unknowns. We solve this system with CG, MINRES, SQMR and MRTR. In figure (2.a), we remark that all these methods converge up to the 5000 iterations where the residual norme is around 10^{-10}, and after that, the CG method diverge while the SQMR, MINRES and MRTR stay with there convergence level. In figure 2.b, the SSOR preconditioner is introduced; we can see

the convergence acceleration for all methods up to the 200 iterations. In this figure we can see that their behaviour is the same as in the case without preconditioner.

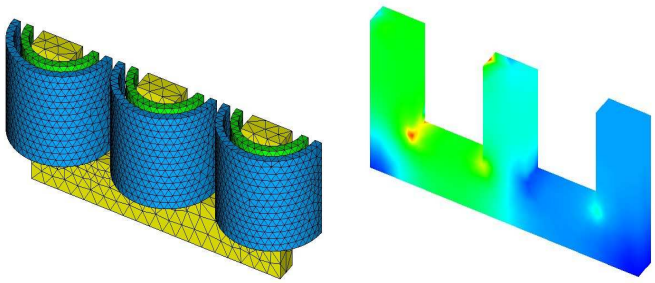

Fig. 1. (a): Mesh of the transformer (b): The flux density in the iron core of the transformer

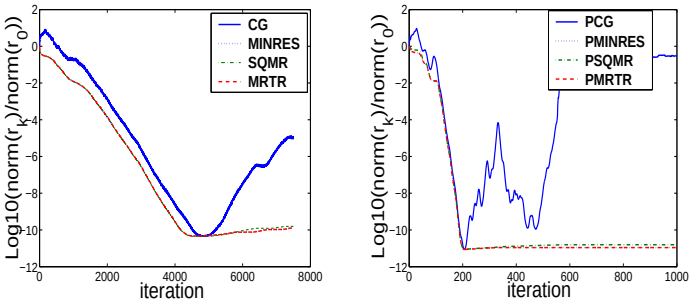

Fig. 2. (a):Without preconditioner (b): SSOR preconditioner

As example of eddy current problem, the Workshop N^{o}8 (Non Destructive Testing problem) is modeled with the $\mathbf{A}$-φ formulation. The mesh and eddy current density are presented respectively on figures (3.a) and (3.b)

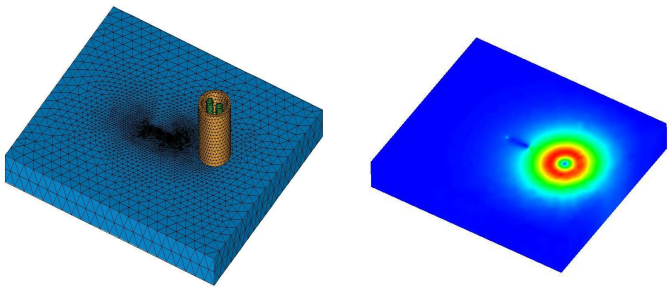

Fig. 3. (a): Mesh of NDT problem (b):Current Density in the conductor

Here we solve a singular system with 212545 unknowns. Without preconditioner, the CG method does not converge even after 5000 iterations. The MRTR, SQMR and MINRES converge slowly, and reaches 10^{-6} at the 5000 iterations (figure 4.a). With SSOR preconditioner, the convergence of all these methods accelerate to about 200 iterations with $\frac{\|r_k\|}{\|r_0\|} \simeq 0,5\ 10^{-13}$, and after that, the CG method diverge while other methods stagne, see figure (4.b).

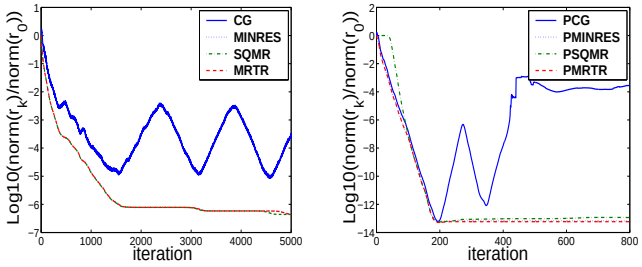

Fig. 4. (a):Without preconditioner (b): SSOR preconditioner

We note that in this two examples, the MRTR, MINRES and SQMR methods have the same behaviour, and give the same result x_{MR}. However, the solution x_{CG} associated to the minimal residual of CG, is different than x_{MR}, and $x_{MR} - x_{CG}$ is in the Kernel of the system matrix.

V. Conclusion

In the case of the singular systems which have the right hand side in the range of the matrix; there exist several solutions. The minimization principle of MINRES, SQMR and MRTR guarantees a regular and smoother convergence than CG. This result is still available when we add a preconditioner to accelerate the convergence. These methods become more interesting than the CG method.

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